Benign approximations, superspeedability, and randomness

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Based on joint works with Peter Hertling, Philip Janicki, Wolfgang Merkle, and Frank Stephan

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Motivation

Left-computable and computable reals

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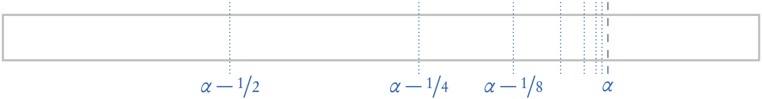

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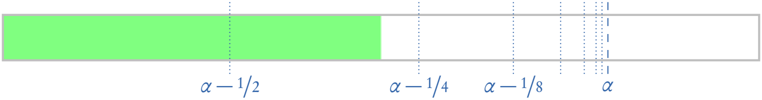

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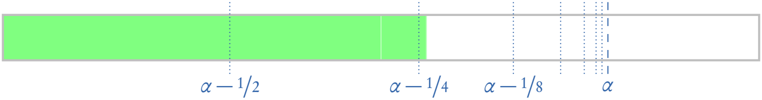

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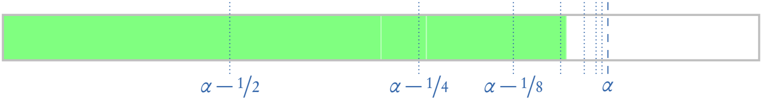

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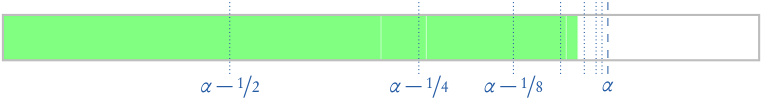

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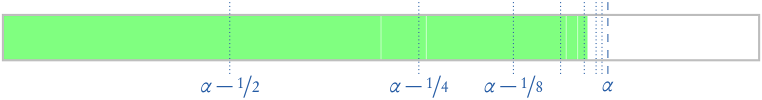

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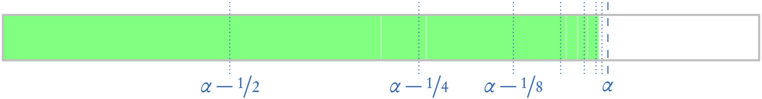

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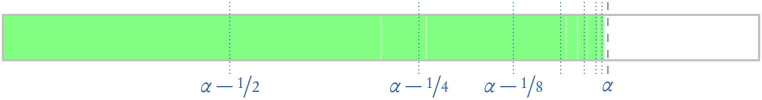

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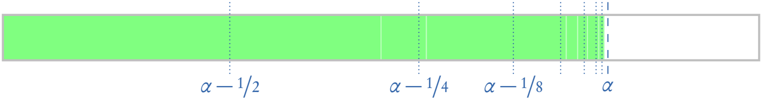

4. So far, so trivial.

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5. Finally, we inquire into the relationship with randomness.

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Three types of benign approximations

Speedability

1 **Definition (Merkle & Titov).** α is *speedable* if there is

- a $\rho \in (0,1)$ and
- a computable left-approximation $(a_n)_n$ of α

such that there are infinitely many $n \in \mathbb{N}$ with

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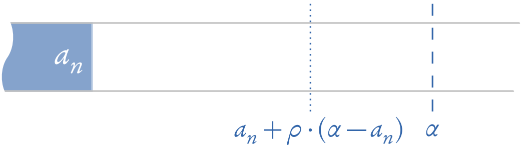

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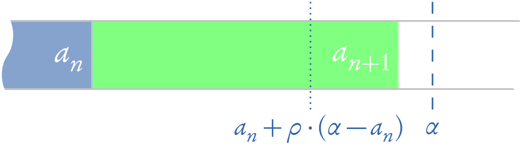

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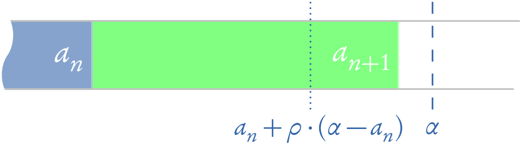

2 **Theorem (Merkle & Titov).** Any $\rho \in (0,1)$ works equally.

(But you need to nonuniformly replace the approximation by another one.)

Speedability & randomness

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2. **Question (Merkle & Titov).**
   - Does the inverse hold?
   - **That is:** Among the left-computables, are the randoms characterized by their non-speedability?

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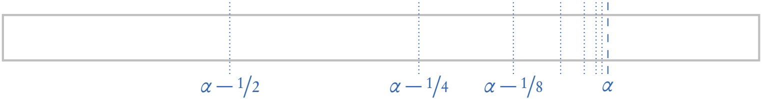

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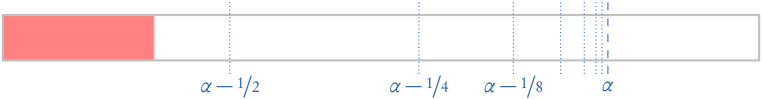

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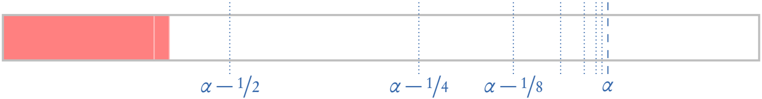

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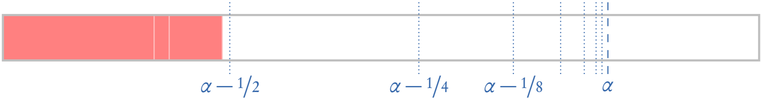

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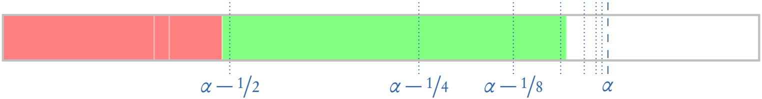

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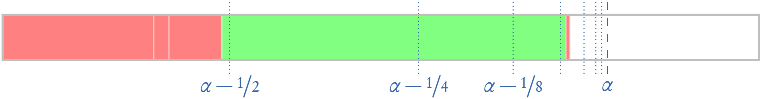

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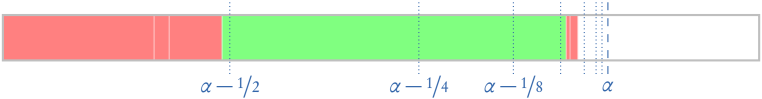

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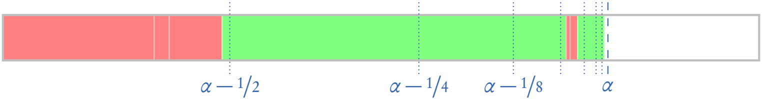

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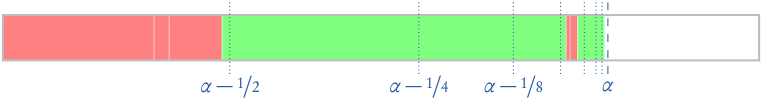

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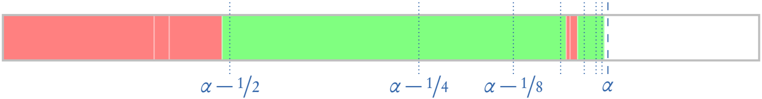

3. **Intuition.**
   - As in general left-computability, the approximation can "dawdle" arbitrarily, *but* infinitely often it must "catch up" to how fast computable numbers can be approximated.
   - Obviously, (in general) we do not know when these good moments occur; in case we do, α is again computable.

Approximations that catch up

1. Note that regaining approximability seems like a really natural notion. We expected to find previous work on this, but it seems no one looked at them before.
2. Thus, with Peter Hertling, we studied many of their properties. Let's mention only the ones most relevant for this talk.

Some selected properties

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4. **Question.** Does it coincide with K-triviality?

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   - Iterate. □

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2. Because in order to catch up, a regaining approximation needs to make big jumps. **Question.** Are those moments of speedability?
3. **Answer.** Almost, but not quite. Even a small jump could be the one finally catching up, if a large jump was made previously.

Regaining approximability implies speedability

1. Why might regaining approximability be relevant for us?
2. Because in order to catch up, a regaining approximation needs to make big jumps. **Question.** Are those moments of speedability?
3. **Answer.** Almost, but not quite. Even a small jump could be the one finally catching up, if a large jump was made previously.
4. But still, catching up requires making some big jump *somewhere*, and we can prove the following statement as a consequence.
5. **Proposition.** Every regainingly approximable α is speedable.

The converse is not true

1. **Proposition (Merkle & Titov).** Every left-computable α that is the binary expansion of a c.e. set is speedable.
2. **Theorem.** Not all such α are regainingly approximable.

So what?

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2. Thus, making a number non-regainingly approximable looks easier than making it non-speedable.

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4. Something is still missing, and this brings us to our third notion of benign approximability.

Nearly computable, left-computable numbers

Definition (Hertling & Janicki).

- $f : \mathbb{N} \to \mathbb{N}$ is a *modulus of convergence* of $(a_n)_n$ if for all $n \in \mathbb{N}$, and all $m \geq f(n)$, we have $|\alpha - a_m| < 2^{-n}$, where $\alpha = \lim a_n$.

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  (This is a special case for left-computables; good enough for us.)

2 **Theorem (Downey & LaForte; reformulated).** There are non-computable, left-computable, nearly computable numbers.

(In their original formulation, they showed the existence of a non-computable, left-computable number all of whose *presentations* via prefix-free c.e. sets are computable.)

3 **Intuition.** Knowing computable upper bounds on the *size* of individual jumps that may still be made doesn't "computably determine" their total *sum*.

How does this help answer the open question?

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1. **Recall:** We have that
   speedability implies regaining approximability.

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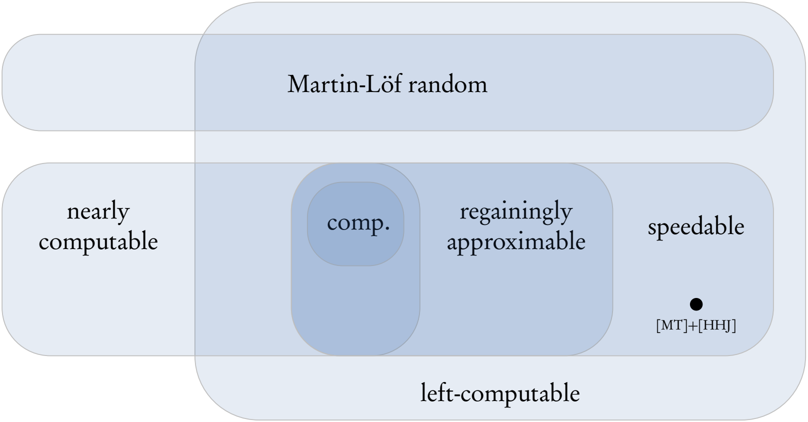
3. **Question.** Can we do so while avoiding randomness, to answer the open question of Merkle & Titov? **Yes!**
4. **Theorem (Stephan & Wu; reformulated).** Left-computable nearly computable numbers cannot be Martin-Löf random.

What we know so far & what remains to do



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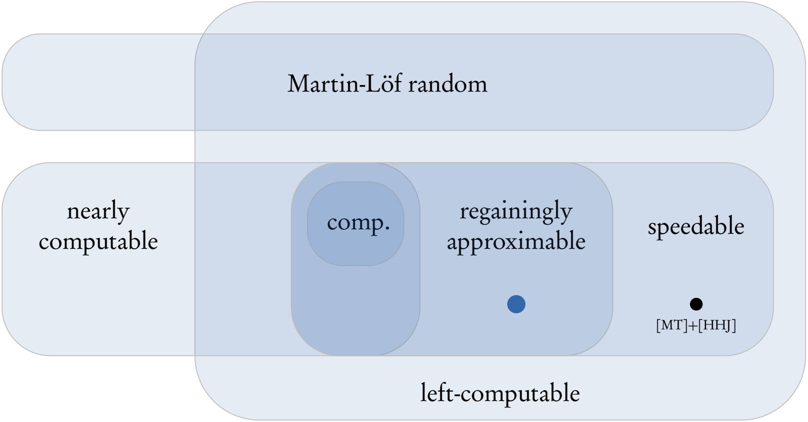

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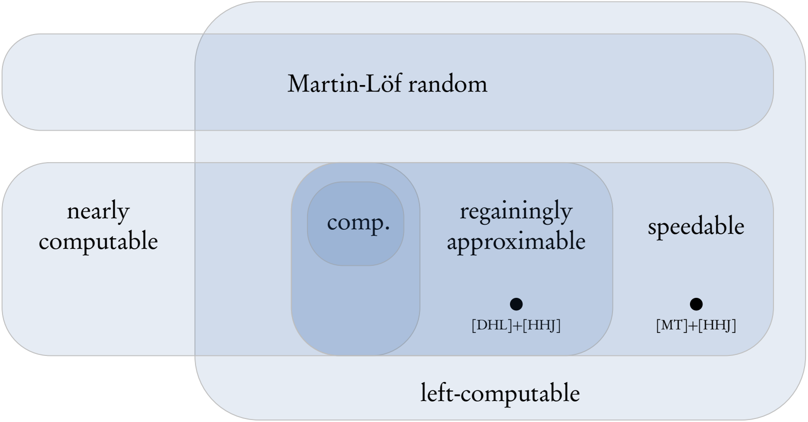

1 **Corollary.** Existence follows from a result of Downey, Hirschfelt and LaForte, combined with ours. (Details omitted.)

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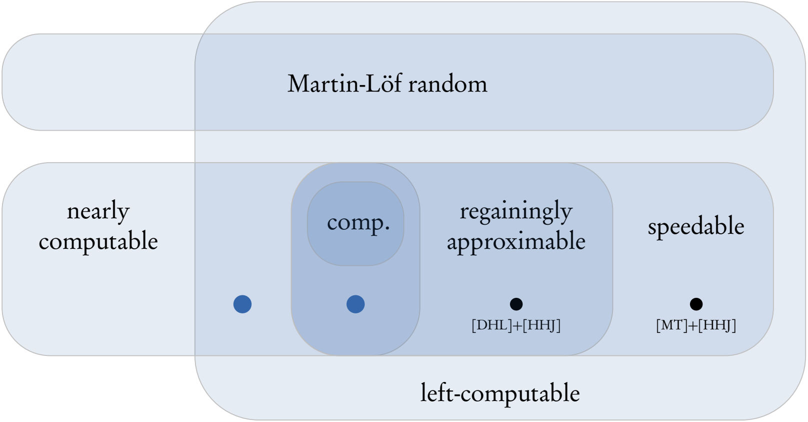

2 It remains to show that these two elements exist.

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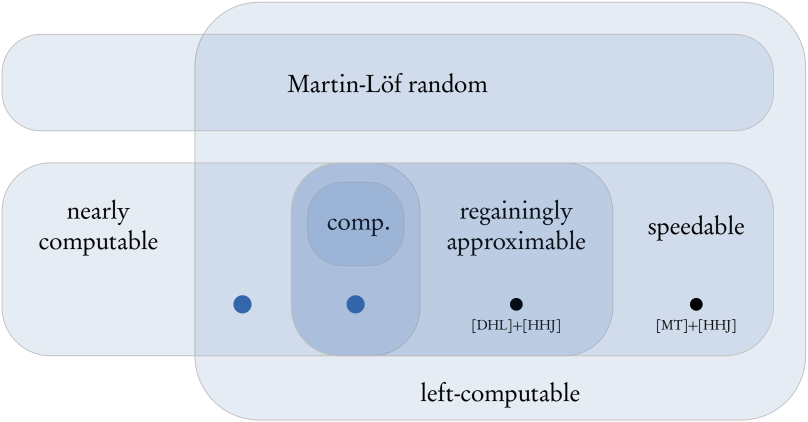

2. It remains to show that these two elements exist.
3. The left one then answers the open question negatively.

3 Constructing the missing points

Constructing the missing points

1. The proofs are inspired by Downey & LaForte's proof that non-trivial nearly computable numbers do exist.
2. But they are more complex because we need to satisfy more and more complex requirements.
3. We can only hint at some of the main ideas here.

Compatibility of the notions

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   $$\mathcal{N}_e\colon\ \varphi_e \text{ total and increasing} \Rightarrow (\exists m \in \mathbb{N})\ \alpha - a_{\varphi_e(m)} \geq 2^{-m}.$$

5. **For near computability:**

   $$\mathcal{P}_e\colon\ \varphi_e \text{ total and increasing} \Rightarrow$$
   $$(a_{\varphi_e(t+1)} - a_{\varphi_e(t)})_t \text{ converges computably to } 0.$$

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1. These two types of requirements seem to be in conflict:
   - The left-approximation of α we construct may need to satisfy negative requirements by performing large jumps rather late.
   - But for positive requirements, we need to commit at certain points to never again making jumps larger than some bound.

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2. We need to carefully balance out these necessities:
   - If a low priority strategy wants to make a large jump, but can't due to a higher priority commitment, then that jump is divided into smaller jumps that are then scheduled for later execution.
   - Even stricter commitments by even higher priority strategies might lead to even further splitting.
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   - Even stricter commitments by even higher priority strategies might lead to even further splitting.
   - A negative requirement is satisfied once all corresponding small jumps have been executed.
3. Our task is to ensure that all required jumps are executed eventually. This is hard because "φ_e is total and increasing" is a non-computable property, necessitating the use of infinite injury.

Compatibility of the notions

1. To also achieve regaining approximability, we want to use a similar idea as above when we were copying Ω:
   - At certain times, we want to scale down the entire game, so that the sum of all future jumps is sufficiently upper bounded.
2. This needs to be done very carefully, because the mechanism discussed on the last slide obliges us to make jumps of defined sizes waiting in queues for their turn. No scaling allowed!
3. Our way out is to make sure that there exist infinitely many *cut-off stages* when the queues are (in some sense) empty enough. □

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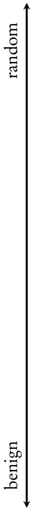
4. The construction is similar to the previous one, except
   - the infinitary negative requirements need different timing and a different initialisation strategy, and
   - this time we need not ensure the existence of cut-off stages, making the verification significantly easier. □

4 A different perspective

Benignness versus randomness

Benignness versus randomness



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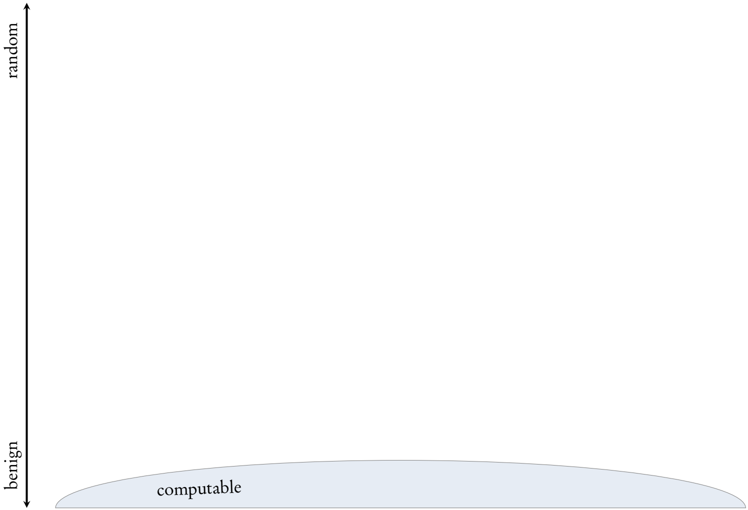

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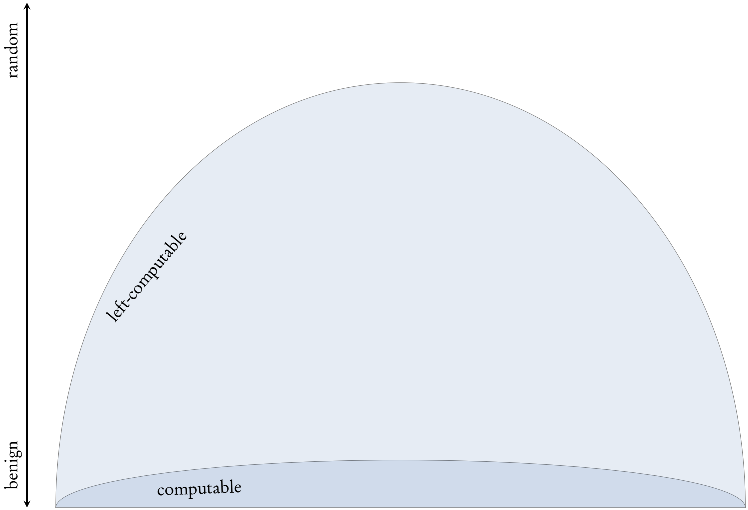

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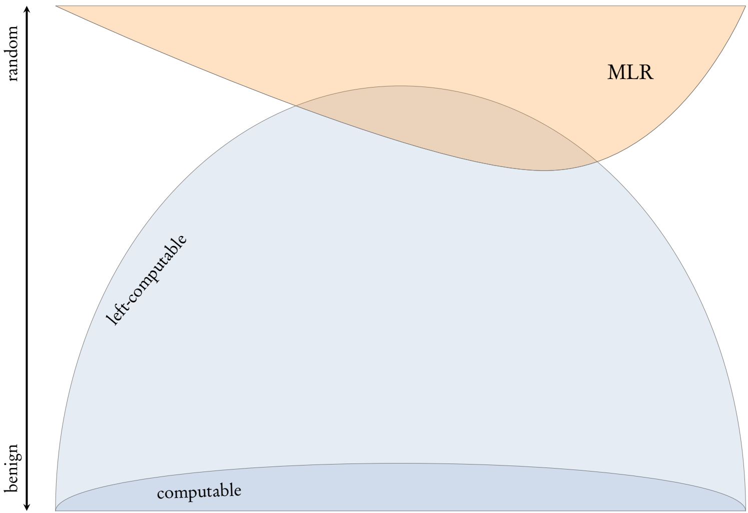

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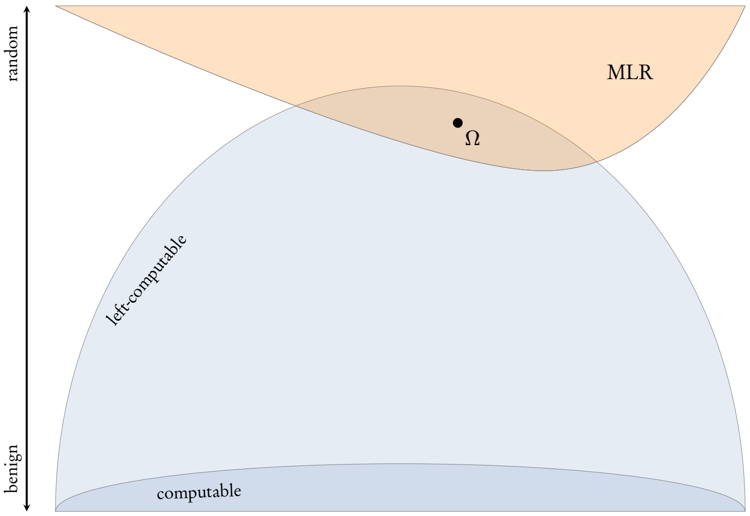

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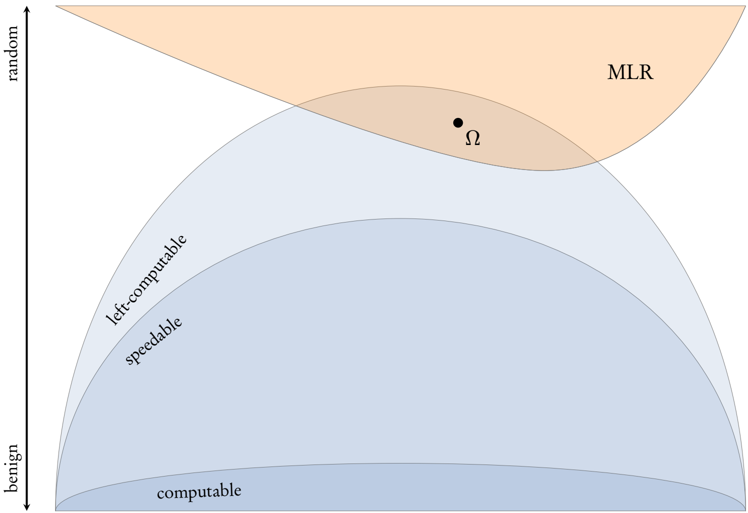

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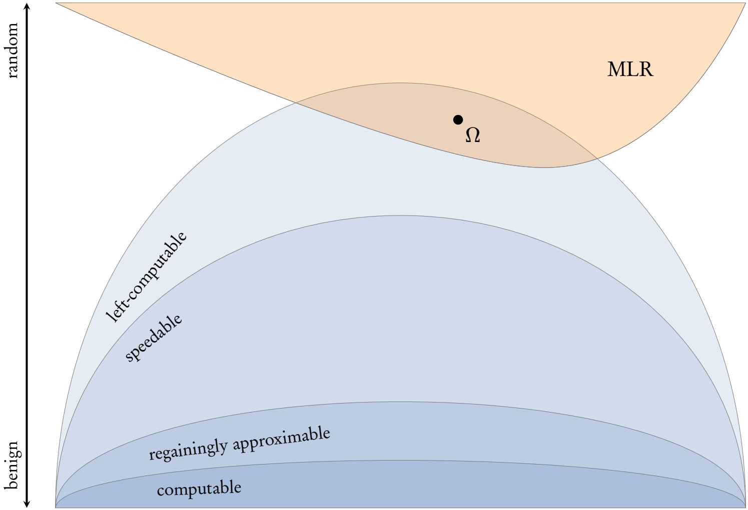

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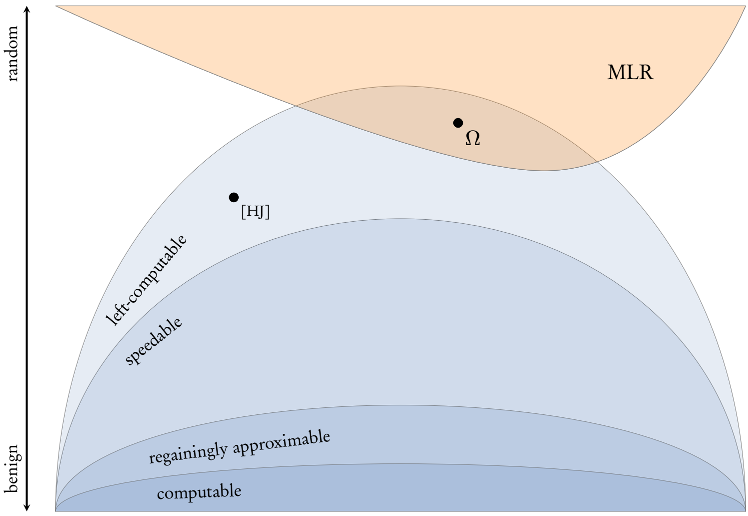

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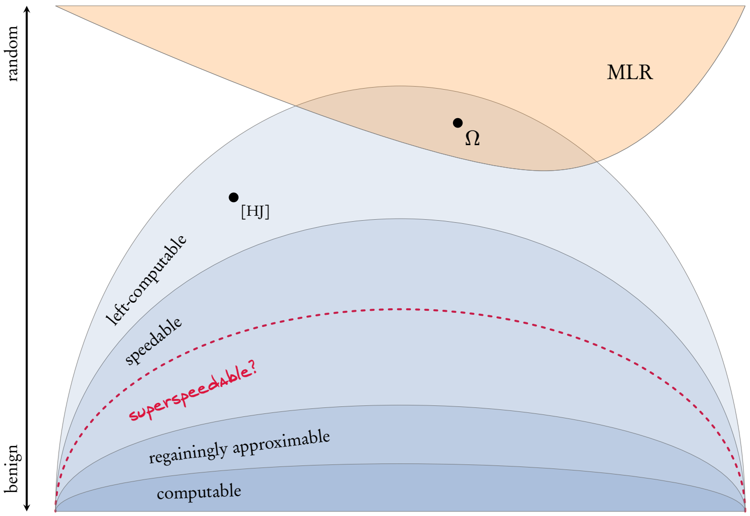

1 **Question (Barmpalias).** Do all speedables have a *single* approximation whose ρ goes to 1? Or is that a smaller set?

Superspeedability?

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$$\limsup_{n\to\infty} \frac{a_{n+1} - a_n}{\alpha - a_n} = 1.$$
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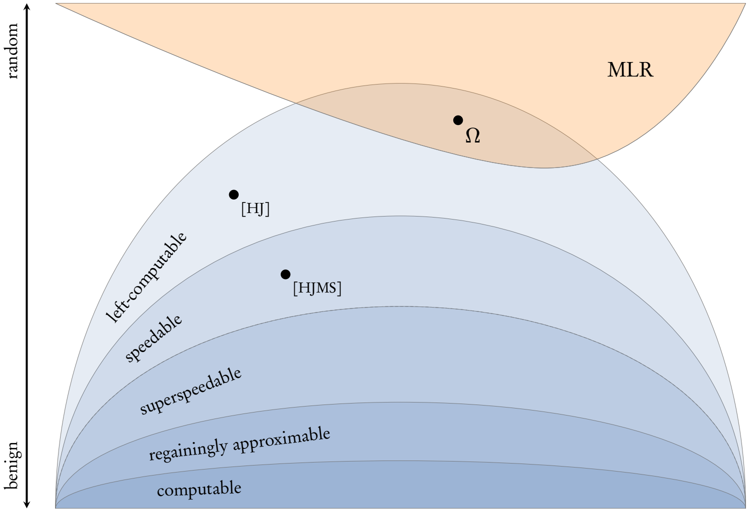
2. **Question (Barmpalias).** Is speedable = superspeedable?
3. **Theorem (Titov?).** For left-computable numbers,
not immune implies speedable.
4. **Theorem.** There exists a left-computable number that is
not immune and not superspeedable.

Benignness versus randomness



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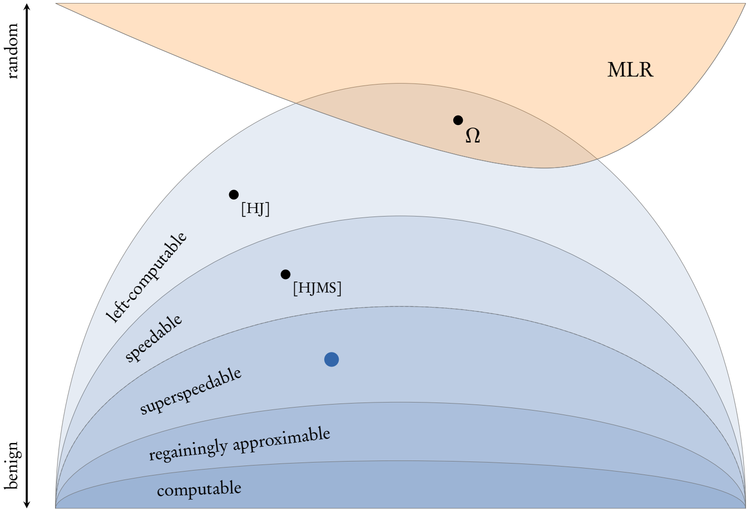

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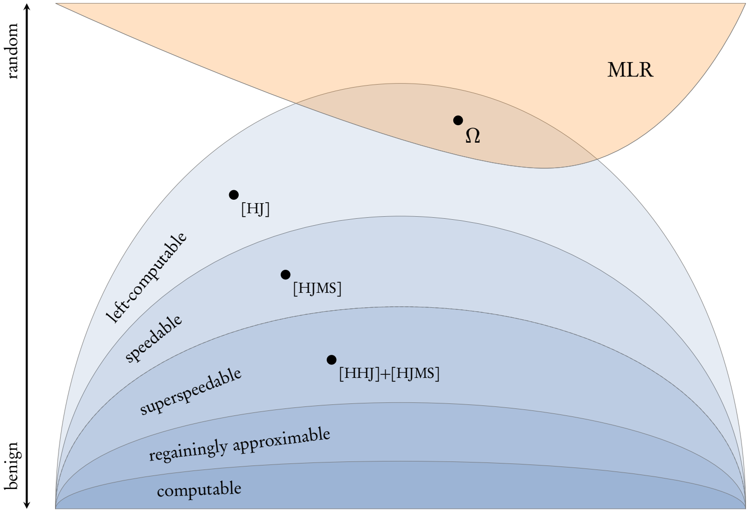

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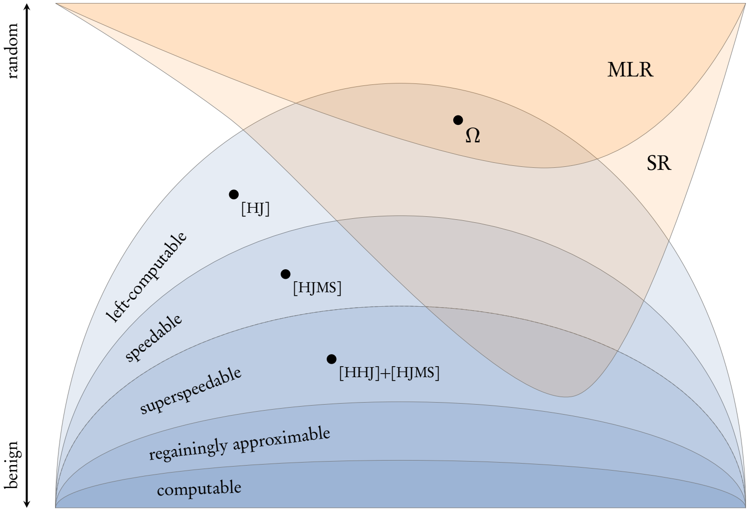

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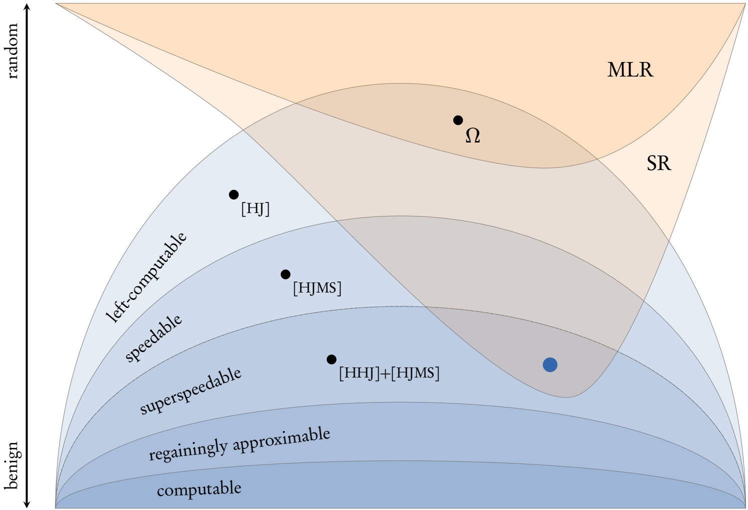

A superspeedable Schnorr random

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4. Define Ω_A bitwise via

$$\Omega_A(n) := \begin{cases} \Omega(m) & \text{if } n = p_A(m), \\ 0 & \text{else;} \end{cases}$$

   (That is, all bits of Ω are there, but they are stored at those positions that are in A.)

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Proof sketch.

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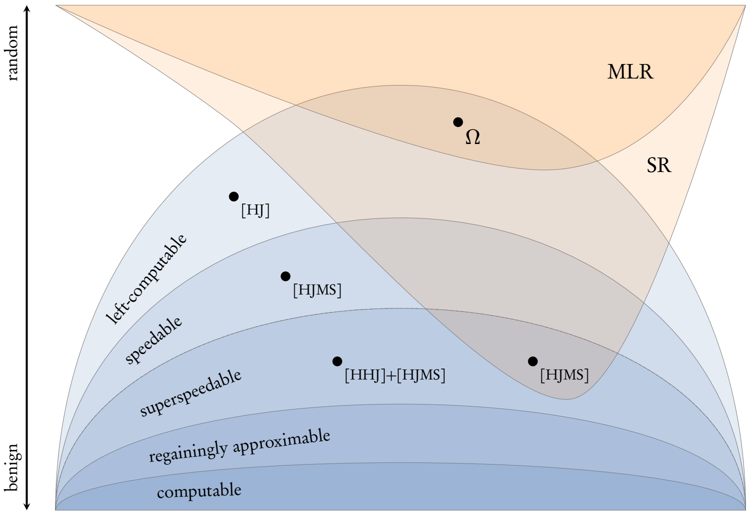
   - By Kolmogorov complexity arguments, c.e. A settles much faster than random Ω. Thus, at the time of the match, d' knows $\overline{A}$.
   - Then omitting all bets on $\overline{A}$ leads to d' betting on Ω as d would have bet on the corresponding bits encoded in Ω_A.
   - As $\overline{A}$ is incomputably thin, but d had a *computable* winning speed, an infinite portion of its winnings must have originated from the bits of Ω. Thus d' wins infinitely much on Ω. □

Open questions



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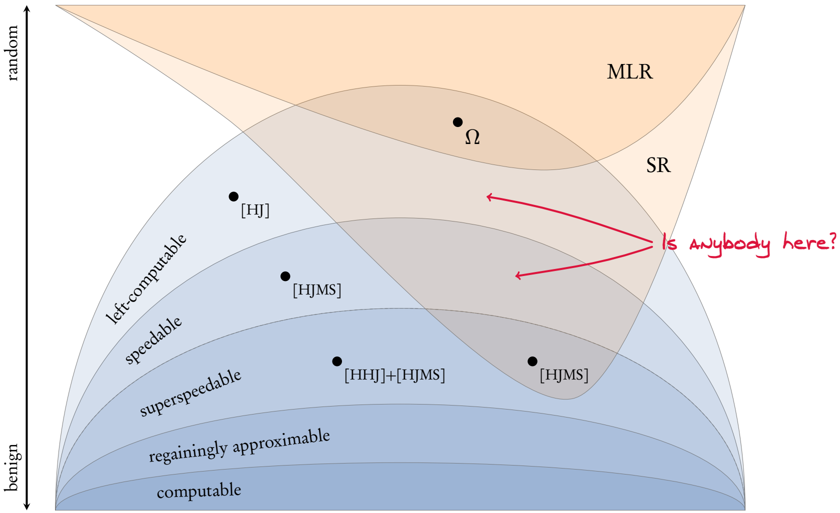
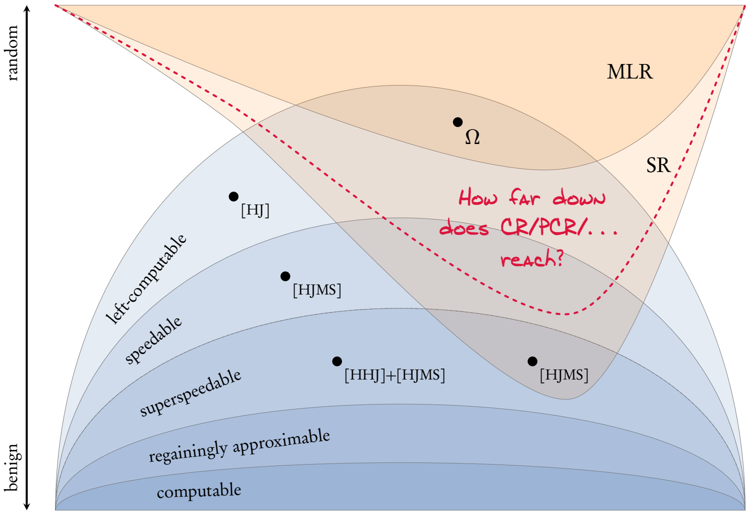


1 **Open question.** Do numbers in the marked fields exist?

(It would be strange if not, but we could not construct any.)

Open questions



2 **Open question.** How benignly approximable can computable randoms, partial computable randoms, weak *s*-randoms,… be?

More open questions

1. **Open question.** Recall that we worked inside the nearly computables to obtain a counterexample to the question of Merkle and Titov. Are there counterexamples outside, too?

   Equivalently: Do the Martin-Löf random numbers, the nearly computable numbers, and the speedable numbers together form a covering of all left-computable numbers?

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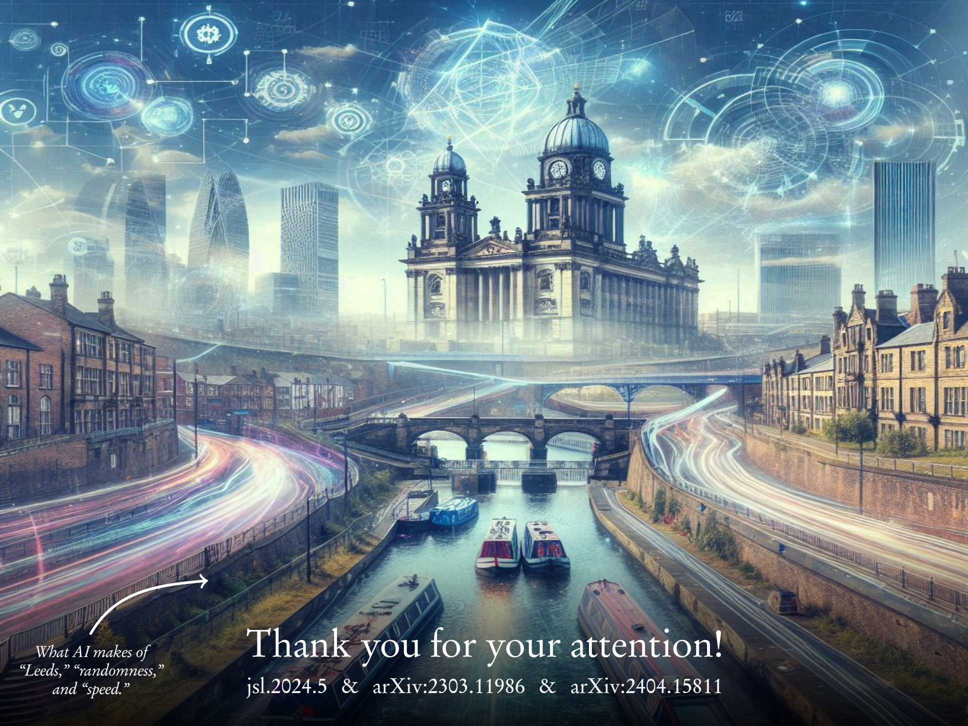
   Equivalently: Do the Martin-Löf random numbers, the nearly computable numbers, and the speedable numbers together form a covering of all left-computable numbers?

2. **Open question.** What are the Weihrauch degrees of incomputable tasks naturally arising in this area? For example,
   - given an approximation witnessing speedability and a desired ρ, find a sequence of stages where ρ is achieved;
   - for a speedable number and a desired ρ, determine another approximation of that number which achieves ρ;
   - for an approximation witnessing regaining approximability, find the n's at which the approximation "catches up;" etc.

What AI makes of "Leeds," "randomness," and "speed."



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Thank you for your attention!

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